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EQUIVALENCE OF TWO MATHEMATICAL PROGRAMS WITH OPTIMIZATION PROBLEMS IN THE CONSTRAINTS

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July 1973

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ABSTRACT

Two classes of mathematical programs with optimization problems in the constraints have recently been studied by two of the authors. The first class involves mathematical programs in the constraints, and the second class involves max-min problems in the constraints. A computational technique has been developed and shown to be effective in solving problems of the first class. We show here that the computational technique can be applied to problems of the apparently wider second class.

CONTENTS

INTRODUCTION

Reference [1] presents theory, interpretations, and an example of mathematical programs with optimization problems in the constraints. Reference [2] presents a computer program for solving mathematical programs with nonlinear programs in the constraints. A future paper will discuss defense applications of these types of mathematical systems.

When the optimization problems in the constraints are two-sided optimization problems, the overall problem cannot be computationally solved directly by the methods given in Reference [2]. This paper shows how, for a large class of two-sided optimization problems in the constraints, the overall problem can be reformulated to yield one-sided optimization problems in the constraints in a form suitable for solution using the methods given in Reference [2].

PROBLEM DESCRIPTION

In Reference [1] Bracken and McGill discuss the following problems:

A) $\underset{\xi}{\text{minimize}}\ \Phi(\xi)$

subject to

$$\xi \in \Xi,$$

$$\min_{\nu^i \in N^i} \gamma^i(\xi, \nu^i) \geq 0, \qquad i = 1, \ldots, m$$

and

B) $\underset{x}{\text{minimize}}\ f(x)$

subject to

$$x \in X,$$

$$\max_{u^i \in U^i(x)} \ \min_{v^i \in V^i} g^i(x, u^i, v^i) \geq 0 \quad , \ i = 1, \ldots, m.$$

These are shown to be convex programs under appropriate assumptions on the functions Φ, γ^i, f, g^i and the sets Ξ, N^i, X, $U^i(x)$, and V^i. In Reference [2], the same authors describe a computational procedure which has been implemented for solving convex programs of type A.

The constraint set for problems of type B was noted in Reference [1] to be equivalent to the set

$$\{x \in X: \min_{v^i \in V^i} g^i(x,u^i,v^i) \geq 0 \text{ for some } u^i \in U^i(x),\ i=1,\ldots,m\}.$$

This paper expands upon this equivalence and shows that problems of type B can be recast into problems of type A and solved by the procedure given in Reference [2].

THEORY

Define the following problem:

C) $$\min_{(x,\ u^1,\ \ldots,\ u^m)} f(x)$$

subject to

$$x \in X,$$

$$u^i \in U^i(x) \qquad i = 1, \ldots, m,$$

$$\min_{v^i \in V^i} g^i(x, u^i, v^i) \geq 0, \qquad i = 1, \ldots, m.$$

Note that problem C has the form of problem A when the vector ξ is replaced by the vector $(x, u^1, \ldots, u^m)$. Also note that we make no convexity or concavity assumptions in the theorem which follows. The computational procedure requires such, however, and the reader is referred to References [1] and [2] for further details. We assume that all of the max or min operations used in the theorem exist. This assumption is valid, for example, when the functions $f, g^1, \ldots, g^m$ are continuous and the sets X, $V^i (i = 1, \ldots, m)$, and $\{(x, u^1, \ldots, u^m)\colon\ u^i \in U^i(x),\ i = 1, \ldots, m\}$ are compact.

THEOREM. Problem B is equivalent to problem C.

Proof. Let $\bar{x} \in X$ be a feasible point of problem B. Then there are vectors $\bar{u}^1, \ldots, \bar{u}^m$ such that $\bar{u}^i$ solves

$$\max_{u^i \in U^i(\bar{x})} \quad \min_{v^i \in V^i} \; g^i(\bar{x}, u^i, v^i)$$

and, moreover, this quantity is nonnegative. Hence,

$$\bar{u}^i \in U^i(\bar{x})$$

and

$$\min_{v^i \in V^i} \; g^i(\bar{x}, \bar{u}^i, v^i) \geq 0,$$

i.e., $(\bar{x}, \bar{u}^1, \ldots, \bar{u}^m)$ is feasible to C.

Now let $(\bar{x}, \bar{u}^1, \ldots, \bar{u}^m)$ be feasible to problem C. Then $\bar{x} \in X$, $\bar{u}^i \in U^i(\bar{x})$ $(i = 1, \ldots, m)$, and

$$h_i(\bar{x}, \bar{u}^i) = \min_{v^i \in V^i} \; g^i(\bar{x}, \bar{u}^i, v^i) \geq 0 \qquad i = 1, \ldots, m.$$

Thus

$$\max_{u^i \in U^i(\bar{x})} \; h_i(\bar{x}, \bar{u}^i) \geq 0, \qquad i = 1, \ldots, m$$

so that, in fact, $\bar{x}$ is feasible to B .

Since the objective function $f(x)$ is common to both problems, it follows that problems B and C have a common optimal value, completing the proof.

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